# Sensitive regions and optimal perturbations in the floating zone using the adjoint system

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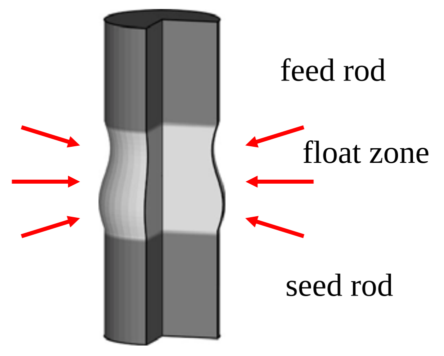
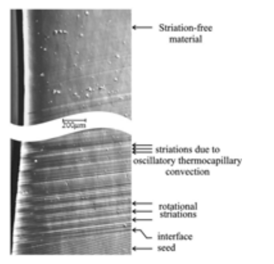
**Floating-Zone** is a crucible free technique to make growing high quality monocrystals. The polycrystaline feed rod changes its structure to monocrystaline during its resolidification on the seed rod after traveling through the laterally heated float-zone.



**Drawback**: oscillating flows can induce structural defects in the, supposed, monocrystal.



A.Cröll et al, *Floating zone and floating-solution-zone growth of GaSb under microgravity*, J. of Crystal Growth, **191** (1998) 365-376

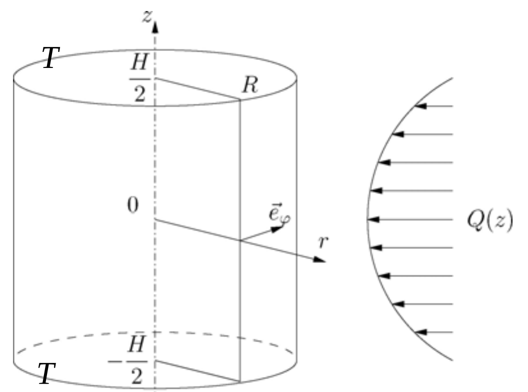
What is the mechanism that makes the flow to oscillate ?

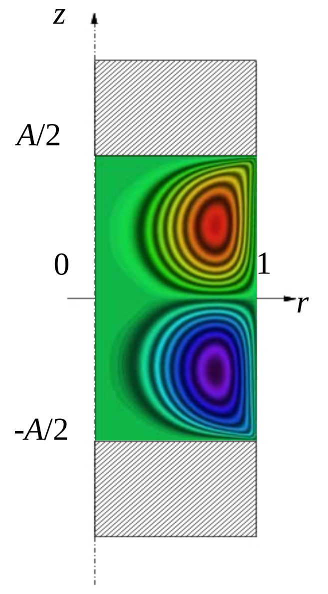
**Objective** : Identification of the mechasim can be done by locating the most sensitive regions with respect to a punctual perturbation.

**Mathematical model:** Navier-Stokes equation and heat equation. Boussinesq approximation

$$\begin{cases} \partial_t \vec{U} + \left(\vec{U}\cdot\vec{\nabla}\right)\vec{U} = -\vec{\nabla}P + Pr\Delta\vec{U} \\ \partial_t T + \left(\vec{U}\cdot\vec{\nabla}\right)T = \Delta T \\ \vec{\nabla}\cdot\vec{U} = 0 \end{cases}$$

$$r=1\begin{cases} U = 0 \\ \partial_r W = -Ma\, f_n(z)\partial_z T \\ \partial_r T = Q(z) \end{cases} \qquad z=\pm\frac{A}{2}\begin{cases} U = 0 \\ W = 0 \\ T = 0 \end{cases}$$





$$Q(z) = \left(1-\left(\frac{2z}{A}\right)^2\right)^2$$

**Parameters:**

| | |
|---|---|
| •Prandtl number *Pr*: | $\vec{V}$. Momentum diffusivity/ $\vec{V}$. Heat diffusivity |
| •Marangoni number *Ma*: | $\vec{V}$. Thermocapillary convection / $\vec{V}$. Heat diffusivity |
| •*A*: | Aspect Ratio, fixed at 2 |
| •*n*: | Regularisation parameter |

**Numerical method:**
- Pseudo-spectral method with Tchebycheff polynomials
- Gauss-Radau grid along radial axis
- Gauss-Lobatto grid along axial axis
- Regularizing function $f_n(z) = \left(1-z^{2n}\right)^2$ is introduced to avoid singularity problem at the corners

**Linearised equations:** $^t\left(\vec{U},T\right) = {}^t\underbrace{\left(\vec{U}_0,T_0\right)}_{\text{steady state}} + \underbrace{{}^t(\vec{u},\theta)}_{\text{perturbation}}$

$$\begin{cases} \partial_t \vec{u} + \left(\vec{U}_0\cdot\vec{\nabla}\right)\vec{u} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{U}_0 = -\vec{\nabla}p + Pr\Delta\vec{u} \\ \partial_t \theta + \left(\vec{U}_0\cdot\vec{\nabla}\right)\theta + \left(\vec{u}\cdot\vec{\nabla}\right)T_0 = \Delta\theta \\ \vec{\nabla}\cdot\vec{u} = 0 \end{cases}$$

$$r=1\begin{cases} u = 0 \\ \partial_r w = -Ma\, f_n(z)\partial_z \theta \\ \partial_r \theta = 0 \end{cases} \qquad z=\pm\frac{A}{2}\begin{cases} u = 0 \\ w = 0 \\ \theta = 0 \end{cases}$$

Eigenvectors $^t(\vec{u}_i,\theta_i)$ and associated eigenvalues $\lambda_i$ are determined with an Arnoldi method.

Decomposition of the perturbation on the eigenbasis is $^t(\vec{u},\theta)(t) = \sum_{i=1}^{+\infty} a_i\,{}^t(\vec{u}_i,\theta_i)e^{\lambda_i t}$

When the difference between $\lambda_1$ and $\lambda_2$ is sufficiently high, the first eigenmode stands alone until the incoming of the non-linearities. So, $^t(\vec{u},\theta)(t) \simeq a_1\,{}^t(\vec{u}_1,\theta_1)e^{\lambda_1 t}$

Introduction of a scalar product to find $a_1$. Scalar product is used to define the adjoint system:

$$\left\langle {}^t(\vec{u},\theta) \middle| {}^t\left(\vec{\tilde{u}},\tilde{\theta}\right)\right\rangle = \int_{-\frac{A}{2}}^{\frac{A}{2}}\int_0^1 u\underline{\tilde{u}} + w\underline{\tilde{w}} + \theta\underline{\tilde{\theta}} \quad \mathrm{r dzdr}$$

**Adjoint equations:**

$$\begin{cases} -\partial_t\tilde{u} = \partial_r\tilde{p} + \left(\vec{U}_0\cdot\vec{\nabla}\right)\tilde{u} - \tilde{u}\partial_r U_0 - \tilde{w}\partial_r W_0 - \tilde{\theta}\partial_r\Theta_0 + Pr\left(\Delta\tilde{u} - \frac{\tilde{u}}{r^2}\right) \\ -\partial_t\tilde{w} = \partial_z\tilde{p} + \left(\vec{U}_0\cdot\vec{\nabla}\right)\tilde{w} - \tilde{u}\partial_z U_0 - \tilde{w}\partial_z W_0 - \tilde{\theta}\partial_z\Theta_0 + Pr\Delta\tilde{w} \\ -\partial_t\tilde{\theta} = \left(\vec{U}_0\cdot\vec{\nabla}\right)\tilde{\theta} + \Delta\tilde{\theta} \\ \vec{\nabla}\cdot\vec{\tilde{u}} = 0 \end{cases}$$

$$r=1\begin{cases} \tilde{u} = 0 \\ \partial_r\tilde{w} = 0 \\ \partial_r\tilde{\theta} = PrMa\partial_z\tilde{w}\, f_n(z) \end{cases} \qquad z=\pm\frac{A}{2}\begin{cases} \vec{\tilde{u}} = \vec{0} \\ \tilde{\theta} = 0 \end{cases}$$

The eigenvalues of the adjoint system are opposite and conjugate to the eigenvalues of the linearised system. The corresponding eigenvectors $^t\left(\vec{\tilde{u}}_i,\tilde{\theta}_i\right)$ are such that :

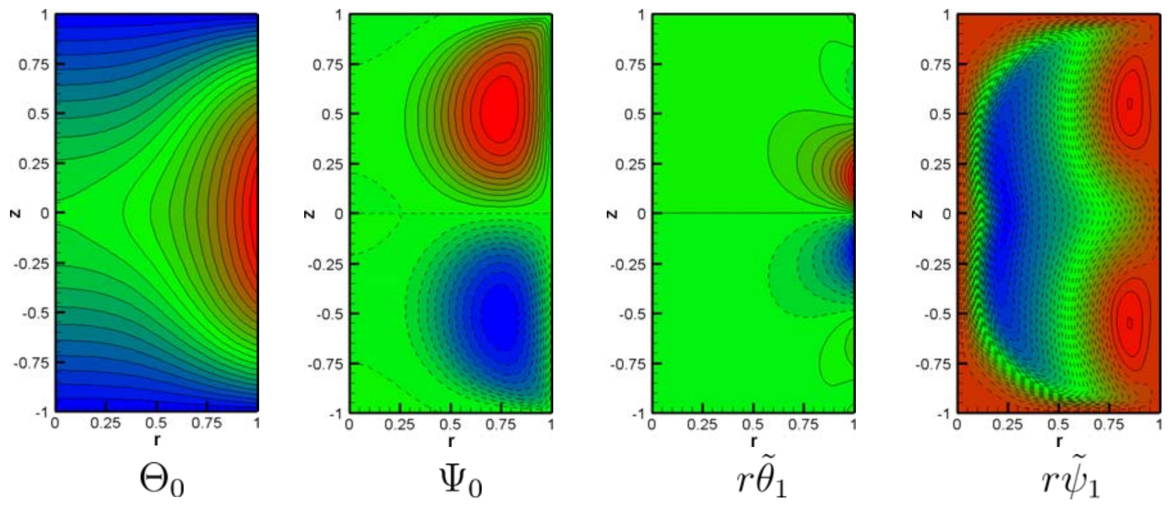
$$\left\langle {}^t(\vec{u}_i,\theta_i) \middle| {}^t\left(\vec{\tilde{u}}_j,\tilde{\theta}_j\right)\right\rangle = \delta_{ij}$$

For a given initial perturbation, we can determine the value of $a_1$

$$\left\langle {}^t(\vec{u},\theta)(t=0) \middle| {}^t\left(\vec{\tilde{u}}_1,\tilde{\theta}_1\right)\right\rangle = a1$$

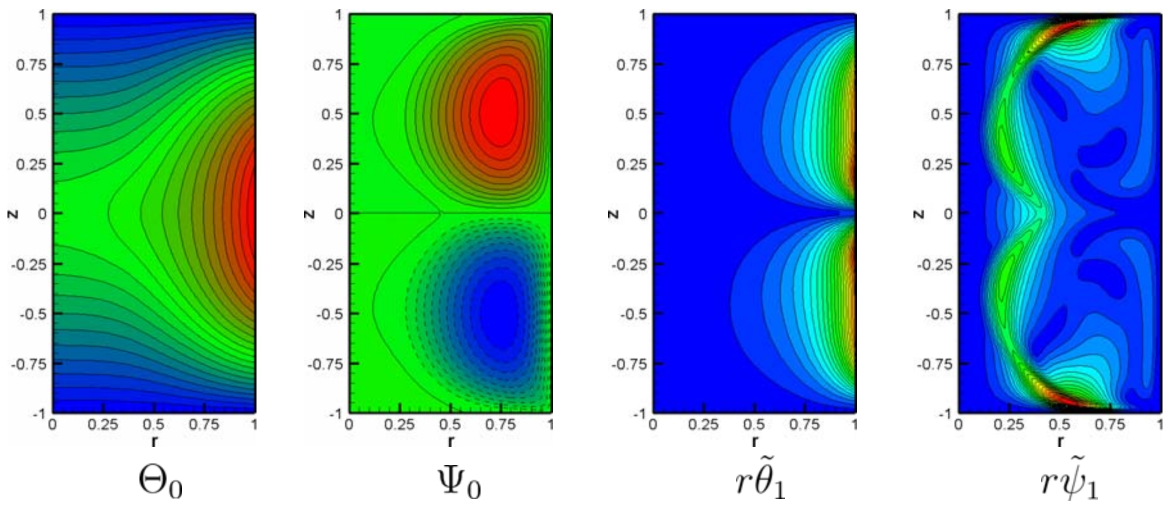
**Results:**

**Stationnary perturbation**
**Pr=0.01 and Ma=106**

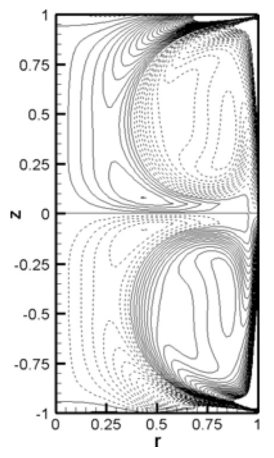


$\Theta_0$ $\Psi_0$ $r\tilde{\theta}_1$ $r\tilde{\psi}_1$

**Oscillatory perturbation**
**Pr=0.002 and Ma=130**



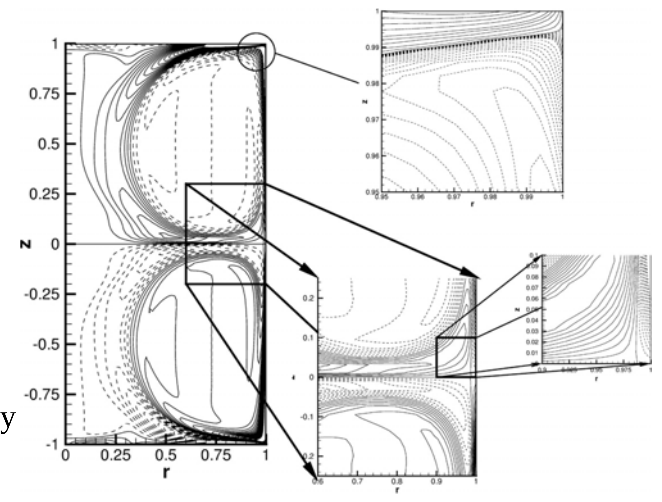
$\Theta_0$ $\Psi_0$ $r\tilde{\theta}_1$ $r\tilde{\psi}_1$



vorticity

In the most sensitive region of the flow with respect to punctual temperature perturbation there is a vorticity structure with a maximum in the radial direction. It can be indentified as an evidence of the Fjørtøft stability criteria for an invicid flow.

Optimal punctual vorticity perturbation for stationnary flow is the most efficient near the mid-plane and the axis whereas for the unstationnary perturbation is more efficient near the walls. Energy analysis shows that those regions are located upstream of regions with high energy growth rate for the perturbation.



vorticity

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